Universal Equations for Continuously Variable 4-Digit Symmetric NACA Airfoils

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1 Background

The NACA family of airfoil geometry is usually conveniently generated through an airfoil plotter available online, which generates a set of data points through which a smooth curve could be fitted in a CAD programme or alike. The curve may then be used for further purposes. This approach is widely used in less demanding modelling scenarios.

However, in parametric modelling where much freedom to modify the model through the control of parameters is desired, it is required that the geometry of the airfoil can be varied parametrically within the CAD model file, which may then be passed onto other software programmes such as FEA packages or optimisers without further external reference to online resources. The ability to define the airfoil geometry mathematically within the CAD programme also provides improved robustness for the model when other parameters are being varied, without potential pitfalls such as unrestrained splines assuming reversed curvature, or small gaps originating due to a lack of fit, giving rise to errors.

Most CAD packages do not allow the complicated algorithm of an airfoil plotter to be implemented. Instead, they demand explicit equations by which the curve may be drawn. A good example of this would be SOLIDWORKS. CREO Parametric, as a more advanced software package, does allow the implementation of simple mathematical operations, but still, they lack the ability to solve for the numerical coefficients required to plot an airfoil. Therefore, it is necessary to have a set of formulae to enable the determination of airfoil geometry by the substitution of desired properties (e.g. maximum thickness location) only.

This article aims to present a set of explicit parametric formulae which governs the shape of a family of NACA symmetric airfoil variants. By *explicit parametric*, it is meant that the equation can be determined by a straightforward substitution of the control parameters.

2 Source Definition

According to NACA report No.492[1], the NACA 4-digit series airfoil may be modified in shape by changing the y equation into a piecewise function dividing at the point where the thickness of the airfoil reaches a maximum.

$$x = t \quad \text{for } 0 \le t \le 1 \tag{1}$$

$$y = \begin{cases} a_0\sqrt{t} + a_1 t + a_2 t^2 + a_3 t^3 & \text{for } 0 \le t \le m \\ d_0 + d_1(1-t) + d_2(1-t)^2 + d_3(1-t)^3 & \text{for } m < t \le 1 \end{cases} \tag{2}$$

where m is a parameter that controls the fractional position of the maximum thickness point along the chord, and t which varies from 0 to 1, is a parameter for both the x and y equations.

In the appendix of the NACA report, there is a table of the eight coefficients to be used for several discrete shapes of the airfoil. This, however, is not suitable for parametric modelling and optimising purposes, as it is required that the shape can vary continuously. Besides, it is not possible to implement a switch function in most CAD packages to allow for the use of discrete coefficients. Therefore, the following section presents the formulae to determine these coefficients from control parameters, with the rest of this article presenting the mathematical basis on which these formulae are derived.

3 Determination of the Coefficients

The following definitions are used for the rest of this article:

- C chord length of the airfoil section
- m fractional position of the thickest point along the chord
- T fractional thickness of the airfoil
- D fractional thickness of the trailing edge

It should be noted that C is the master scaling factor of the geometry, with all the other parameters being fractional.

Coefficient d_0

$$d_0 = \frac{1}{2}D \tag{3}$$

Coefficient d_1

$$d_1 = -2.5m^4 + 7.1667m^3 - 2.725m^2 + 0.5033m + 0.155 \tag{4}$$

Coefficient d_3

$$d_3 = \frac{-0.2 + (1-m)d_1 + 2d_0}{(1-m)^3} \tag{5}$$

Coefficient d_2

$$d_2 = \frac{-d_1 - 3d_3(1-m)^2}{2(1-m)} \tag{6}$$

Intermediate coefficient R

$$R = \frac{(1-m)^2}{2d_1(1-m) - 0.6 + 6d_0} \tag{7}$$

Coefficient a_0

$$a_0 = 0.2969 \quad \text{for normal nose radius} \tag{8}$$

Intermediate coefficient β

$$\beta = \frac{1}{R} + \frac{a_0}{4(\sqrt{m})^3} \tag{9}$$

Coefficient a_3

$$a_3 = \frac{0.1 - \frac{1}{2}a_0\sqrt{m} + \frac{1}{2}\beta m^2}{m^3} \tag{10}$$

Coefficient a_2

$$a_2 = \frac{1}{2}\beta - 3ma_3 \tag{11}$$

Coefficient a_1

$$a_1 = -\frac{a_0}{2\sqrt{m}} + 3a_3m^2 - \beta m \tag{12}$$

By substituting the above coefficients into Equation (1) and 2, it is possible to obtain a curve for the upper half of the airfoil of unity chord length. In order to obtain the final geometry:

- scale both expressions with the factor C
- scale the y expression with the factor $\frac{T}{0.2}$
- connect the trailing edge with a straight line

4 The Final Equations

With the coefficients determined as described above, and applying the appropriate scaling, the form equations of the modified airfoil may be written in the following form:

$$x = Ct \quad \text{for } 0 \le t \le 1 \tag{13}$$

$$\pm y = \begin{cases} \frac{CT}{0.2}[a_0\sqrt{t} + a_1 t + a_2 t^2 + a_3 t^3] & \text{for } 0 \le t \le m \\ \frac{CT}{0.2}[d_0 + d_1(1-t) + d_2(1-t)^2 + d_3(1-t)^3] & \text{for } m < t \le 1 \end{cases} \tag{14}$$

Alternatively, in the explicit form:

$$\pm y = \begin{cases} \frac{CT}{0.2}[a_0\sqrt{\frac{x}{C}} + a_1\frac{x}{C} + a_2(\frac{x}{C})^2 + a_3(\frac{x}{C})^3] & \text{for } 0 \le x \le mC \\ \frac{CT}{0.2}[d_0 + d_1(1-\frac{x}{C}) + d_2(1-\frac{x}{C})^2 + d_3(1-\frac{x}{C})^3] & \text{for } mC < x \le C \end{cases} \tag{15}$$

5 Derivation of the Coefficients Formulae

This section presents the derivation of the coefficients formulae used above.

Please take note that $x = t$ (from Equation (1)) in the following discussions.

5.1 Properties of the Definition

NACA report No.492 has demanded that the family of the modified symmetric 4-digit airfoils be described in the piecewise form by two third-order polynomials (Equation (2)). It can be immediately seen that this elegant choice of equation form always satisfies the following properties:

1. the curve starts from the origin.
2. the curve terminates at $x = 1$, $y = d_0$, so the trailing edge thickness can be controlled by the coefficient d_0 alone.
3. the curve has an infinite gradient at $x = 0$, i.e.

   $$\lim_{x\to 0}\frac{\mathrm{d}y}{\mathrm{d}x} = \infty$$

   this guarantees that the curve has a round nose unless the coefficient a_0 is set to zero, which would result in a sharp leading edge.

5.2 First Condition of the Equations

Because the form equations are piecewise, in order for the two y equations to match at the point $t = m$, it is necessary to enforce the following:

1. $y = 0.1$ at $t = m$, i.e. the curve reaches a specified value of thickness at the point of $x = m$. Given that the thickness, when scaled, is to be done globally, i.e. as a multiplier in front of the y equations, it is convenient to choose a constant 0.1 here while the scaling can be applied afterwards. This is the reason why the scaling factor as seen in Equation (14) and (15) has a denominator of 0.2.

2. $\left.\frac{\mathrm{d}y}{\mathrm{d}t}\right|_{x=m} = 0$, i.e. the curve reaches a maximum at the point of $x = m$.

Both y equations must satisfy these two conditions.

5.3 Coefficients d_2 and d_3

Consider the second half of the y equation:

$$y = d_0 + d_1(1-t) + d_2(1-t)^2 + d_3(1-t)^3$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = -d_1 - 2d_2(1-t) - 3d_3(1-t)^2$$

By enforcing the condition as described in Section 5.2, the following simultaneous equations are arrived at:

$$\begin{cases} d_0 + d_1(1-m) + d_2(1-m)^2 + d_3(1-m)^3 & = 0.1 \\ -d_1 - 2d_2(1-m) - 3d_3(1-t)^2 & = 0 \end{cases}$$

where $d_0 = \frac{D}{2}$ is specified. It is then possible to arrive at Equation (5) and (6), where d_2 and d_3 are calculated in terms of d_0 and d_1.

5.4 Coefficient d_1

NACA report No.492 has chosen 5 discrete values for coefficient d_1 for reference. The value of d_1 represent the angle of the curve at the trailing edge, as the following equation makes clear:

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=1} = -d_1$$

Value of d_1 must be chosen so that a reversal of curvature is avoided in the after-portion of the curve. By performing a polynomial fit to the data chosen in NACA report No.492, Equation (4) is arrived at. The following table presents the values of d_1 chosen in NACA report No.492, and the respective values as calculated from Equation (4).

Table 1: Value of Coefficient d_1

m	d_1 from NACA	d_1 from Eqn.(4)
0.2	0.200	0.1999936
0.3	0.234	0.2339909
0.4	0.315	0.3149888
0.5	0.465	0.4649875
0.6	0.700	0.6999872

It can be seen that Equation (4) is very precise for the range $0.2 \leq m \leq 0.6$. However, this method is not robust, and the use of this polynomial fitting method is a known defect of the equations presented in this article. Please refer to the next section for further discussion on this issue.

5.5 Coefficient a_0

Please take note that, in Equation (8), the value of coefficient a_0 has been specified. This is no random real number. The coefficient a_0 is related to the radius of the nose of the airfoil.

The formula for radius of curvature of a curve in Cartesian coordinates is:

$$\text{Radius of Curvature} = \frac{(1+y'^2)^{\frac{3}{2}}}{y''}$$
$$\text{where } y' = \frac{\mathrm{d}y}{\mathrm{d}x},\ y'' = \frac{\mathrm{d}^2y}{\mathrm{d}x^2}$$

In this case, it can be determined that:

$$y' = \frac{a_0}{2\sqrt{x}} + a_1 + 2a_2x + 3a_3x^2$$
$$y'' = -\frac{a_0}{4(\sqrt{x})^3} + 2a_2 + 6a_3x$$

Substituting into the radius of curvature formula gives:

$$\lim_{x \to 0} R = \frac{{a_0}^2}{2}$$

Therefore, it is further possible to define a fifth control parameter to control the nose radius, from which a_0 may be determined, and the formulae presented above will work for a range of values for a_0. However, this is out of the scope of this article. It must be noted that a_0 shall not take a negative value.

5.6 Second Condition of the Equations

With the coefficients of the second half of the curve determined, it is now necessary to examine the four coefficients of the first half of the curve. The same approach as discussed above may be used, where the First Condition would produce a set of two simultaneous equations.

$$\begin{cases} y = a_0\sqrt{m} + a_1 m + a_2 m^3 + a_3 m^3 & = 0.1 \\ \frac{\mathrm{d}y}{\mathrm{d}t}\Big|_{t=m} = \frac{a_0}{2\sqrt{m}} + a_1 + 2a_2 m + 3a_3 m^2 & = 0 \end{cases}$$

With a_0 determined as described above, there must still be one more condition before these four coefficients can be solved for.

The Second Condition must be enforced: *radius of curvature must match exactly at $t = m$ for the two y equations.*

Radius of curvature of the curve at $x = m$, calculated from the second half, is given by Equation (7), which is derived from the definition, taking note of the following:

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}t}\bigg|_{t=m} &= -d_1 - 2d_2(1-m) - 3d_3(1-m)^2 \\ \frac{\mathrm{d}^2y}{\mathrm{d}t^2}\bigg|_{t=m} &= 2d_2 + 6d_3(1-m) \end{aligned}$$

where d_2 and d_3 should be expressed in terms of d_1 and d_0 by Equation (5) and (6). Note that, since all four d coefficients are now determined, Equation (7) evaluates into a constant R_0.

Radius of curvature of the curve at $x = m$, calculated from the first half, is given by:

$$R = \frac{1}{-\frac{a_0}{4(\sqrt{x})^3} + 2a_2 + 6a_3x} = R_0$$

and this will give the third simultaneous equation required:

$$-\frac{a_0}{4(\sqrt{x})^3} + 2a_2 + 6a_3x = \frac{1}{R_0}$$

It is now possible to solve for a_1, a_2, and a_3, and the result is given by Equation (10), (11), and (12).

6 Defect of these Formulae

The method and results presented above contain a defect, which is related to the determination of the coefficient d_1. It is unclear how was the values presented in NACA report No.492 chosen, and the only condition that d_1 must satisfy is that a reversal of curvature in the second half of the curve must not be allowed.

If the curvature is not to reverse, the sign of the radius of curvature, evaluated at any point between $x = m$ and $x = 1$, must be the same as the sign of R_0 used in the previous section, which is negative. Therefore, this condition can be written as follows:

$$\frac{(1 + y'^2)^{\frac{3}{2}}}{y''} < 0 \quad \text{for } m \leq x \leq 1$$

where $y' = -d_1 - 2d_2(1 - x) - 3d_3(1 - x)^2$, $y'' = 2d_2 + 6d_3(1 - x)$

The condition above is very hard to be solved for analytically, but it should yield an inequality, which the value of d_1 must satisfy. It should be noted that d_2 and d_3 is present in the inequality, *which is related to* d_0. In NACA report No.492, the value of d_1 is only a function of m, and for all airfoils reported, d_0 is kept at a constant value. Therefore, it is unclear if the variation of d_0 will make d_1, which, under the current framework, being only a function of m, violate this condition at some point. Hence, the variation of d_1 must be applied only around the value of 0.002 (which is the value reported in NACA report No.492) in a very cautious manner.

Further work should be done to numerically study the extent to which d_0 can be allowed to vary.

References

[1] John Stack and Albert E. von Doenhoff. *Tests of 16 related airfoils at high speed*. Available at https://ntrs.nasa.gov/search.jsp?R=19930091566